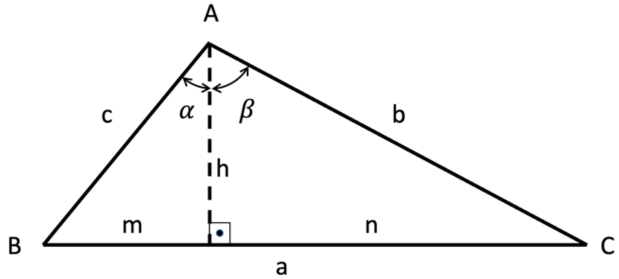
Dedução das fórmulas do seno e cosseno da soma de dois ângulos a partir da lei dos cossenos e dos senos.

Seja o triângulo ABC, tal que:



Note que:

$\cos\alpha = \frac{h}{c}; \cos\beta = \frac{h}{b}$

$sen\ \alpha = \frac{m}{c}; sen\ \beta = \frac{n}{b}$

I) Do teorema dos cossenos, temos:

$a^2 = b^2 + c^2 - 2 \cdot b \cdot c \cdot \cos\hat{A}$

Como $a = m + n$ e $\hat{A} = \alpha + \beta$, então:

$(m+n)^2 = b^2 + c^2 - 2 \cdot b \cdot c \cdot \cos(\alpha+\beta)$

Mas $b^2 = h^2 + n^2$ e $c^2 = h^2 + m^2$. Logo:

$m^2 + 2 \cdot m \cdot n + n^2 = (h^2 + n^2) + (h^2 + m^2) - 2 \cdot b \cdot c \cdot \cos(\alpha+\beta)$

$-m^2 - n^2 + m^2 + 2 \cdot m \cdot n + n^2 = -m^2 - n^2 + 2 \cdot h^2 + n^2 + m^2 - 2 \cdot b \cdot c \cdot \cos(\alpha+\beta)$

$2 \cdot m \cdot n = 2 \cdot h^2 - 2 \cdot b \cdot c \cdot \cos(\alpha+\beta)$

$\frac{2 \cdot m \cdot n}{2} = \frac{2 \cdot (h^2 - b \cdot c \cdot \cos(\alpha+\beta))}{2}$

$m \cdot n = h^2 - 2 \cdot b \cdot c \cdot \cos(\alpha+\beta)$

$m \cdot n - h^2 = -b \cdot c \cdot \cos(\alpha+\beta)$

Isolando o $\cos(\alpha+\beta)$, vem:

$$\cos(\alpha+\beta) = \frac{m \cdot n - h^2}{-b \cdot c} = \frac{-h^2}{-b \cdot c} + \frac{m \cdot n}{-b \cdot c} = \frac{h \cdot h}{b \cdot c} - \frac{m \cdot n}{b \cdot c}$$

Ou seja:

$$\cos(\alpha+\beta) = \frac{h}{c} \cdot \frac{h}{b} - \frac{m}{c} \cdot \frac{n}{b}$$

Assim:

$$\boxed{\cos(\alpha+\beta) = \cos\alpha \cdot \cos\beta - \text{sen}\ \alpha \cdot \text{sen}\ \beta}$$

Exemplo de aplicação:

$$\cos 75° = \cos(45° + 30°) = \cos 45° \cdot \cos 30° - \operatorname{sen} 45° \cdot \operatorname{sen} 30°$$

$$\cos 75° = \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2} \cdot \frac{1}{2} = \frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{4}$$

$$\cos 75° = \frac{\sqrt{6} - \sqrt{2}}{4}$$

II) Do teorema dos senos, temos:

$$\frac{a}{\operatorname{sen} \hat{A}} = \frac{b}{\operatorname{sen} \hat{B}}$$

Mas $\operatorname{sen} \hat{B} = \frac{h}{c}$. Portanto,

$$\frac{a}{\operatorname{sen} \hat{A}} = \frac{b}{\frac{h}{c}} \quad \Longrightarrow \quad \frac{a}{\operatorname{sen} \hat{A}} = \frac{b \cdot c}{h}.$$

Isolando $\operatorname{sen} \hat{A}$, vem:

$$a \cdot h = \mathrm{b} \cdot \mathrm{c} \cdot \operatorname{sen} \hat{A}$$

$$\operatorname{sen} \hat{A} = \frac{a \cdot h}{b \cdot c}.$$

Como $a = m + n$, fazemos:

$$\operatorname{sen} \hat{A} = \frac{(m+n) \cdot h}{b \cdot c} = \frac{m \cdot h + n \cdot h}{b \cdot c} = \frac{m \cdot h}{b \cdot c} + \frac{n \cdot h}{b \cdot c}.$$

Lembrando que $\hat{A} = \alpha + \beta$, concluímos que:

$$\operatorname{sen}(\alpha + \beta) = \frac{m}{c} \cdot \frac{h}{b} + \frac{n}{b} \cdot \frac{h}{c}$$

ou

$$\boxed{\operatorname{sen}(\alpha + \beta) = \operatorname{sen} \alpha \cdot \cos \beta + \operatorname{sen} \beta \cdot \cos \alpha}$$

Exemplo de aplicação:

$$\operatorname{sen} 75° = \operatorname{sen}(45° + 30°) = \operatorname{sen} 45° \cdot \cos 30° + \operatorname{sen} 30° \cdot \cos 45°$$

$$\operatorname{sen} 75° = \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} + \frac{1}{2} \cdot \frac{\sqrt{2}}{2} = \frac{\sqrt{6}}{4} + \frac{\sqrt{2}}{4}$$

$$\operatorname{sen} 75° = \frac{\sqrt{6} + \sqrt{2}}{4}$$